STRICTLY ANTIPODAL SETS*

BY

BRANKO GRÜNBAUM

ABSTRACT

A subset A of E^3 is called strictly antipodal provided that for every pair X_1, X_2 of points of A there is a pair H_1, H_2 of parallel supporting planes of A such that $H_i \cap A = \{X_i\}$. The main result asserts that a strictly antipodal set has at most five points. This strengthens a recent result of Croft [2].

1. **Introduction.** For a convex polyhedron K let $v(K)$ denote the number of vertices of K. If K_1 and K_2 are convex polyhedra it is clear that $v(K_1 + K_2) \leqq v(K_1).v(K_2)$. It is easy to find examples showing that equality may hold for suitable K_1, K_2 in E^3; if $K_1, K_2, \subset E^2$, then

$$v(K_1 + K_2) \leqq v(K_1) + v(K_2).$$

More complicated, and unsolved in the general case, is the following related problem:

If K is a convex polyhedron in E^n, with $v = v(K)$ vertices, how many vertices can $K^* = K + (-K)$ have? It is easily checked that, independent of n, $v(K^*) \leqq v(v-1)$. However, equality in this relation can take place only if v is not too large with respect to n.

Let $f(n)$ denote the maximal v such that there exists an n-dimensional convex polyhedron K with $v = v(K)$ and $v(K^*) = v(v-1)$. It is easily seen that $f(2) = 3$. In Section 2 we shall prove the following result.

THEOREM. $f(3) = 5$.

As easy corollaries we shall obtain (in Section 3) a simple solution of a problem of Erdös [5] recently solved by Croft [2], as well as a number of results on families of translates of convex polyhedra in E^3. Some additional remarks and problems are also given in Section 3.

2. **Proof of the Theorem.** For an arbitrary set $A \subset E^n$ let a pair of points $X_1, X_2 \in A$ be called *strictly antipodal* if there exists a pair H_1, H_2 of (distinct) parallel supporting hyperplanes of A such that $A \cap H_i = \{X_i\}$ for $i = 1,2$. A set A is called strictly antipodal provided ever y two points of A are strictly

Received December 19, 1962

* This research was supported in part by the United States Air Force under Grant No. AF-EOAR 63-63 and monitored by the European Office, Office of Aerospace Research.

antipodal. Let $g(n)$ denote the maximal number of points in a strictly antipodal set $A \subset E^n$. The following assertion is obvious:

LEMMA. The set of vertices of a convex polyhedron K is strictly antipodal if and only if $v(K^*) = v(K)\cdot(v(K) - 1)$.

If A is a strictly antipodal set and K its convex hull, then A coincides with the set of vertices of K. Therefore, the lemma implies that $f(n) = g(n)$.

In order to prove the Theorem it is sufficient to show that the common value of $f(3)$ and of $g(3)$ is 5. Since $f(3) \geqq 5$ (see Section 3) and since every subset of a strictly antipodal set is strictly antipodal, we have only to show that no 6-pointed set in E^3 is strictly antipodal.

Assuming this to be false, let A be a six-pointed strictly antipodal set in E^3, with convex hull K. Because of $f(2) = g(2) = 3$, all the faces of K are triangles. Counting incidences and using Euler's formula it follows at once that K must be a polyhedron of one of the two types represented in Figure 1 by their Schlegel diagrams.

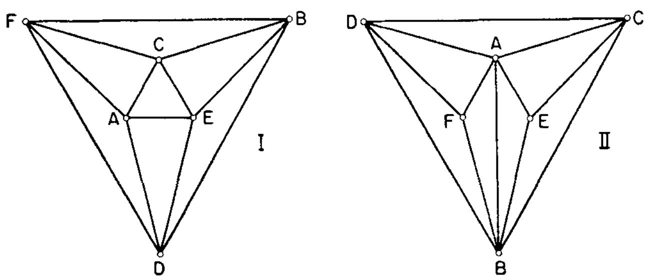

Figure 1

We first note that K cannot have configuration II. Indeed, if there would exist such K, in view of the affine invariance of strict antipodality we could assume that K has the form indicated in Figure 2. Since the segment EF is not an edge of K, $\min\{a,\alpha\} < 1$. Without loss of generality we assume that $a < 1$, and we note

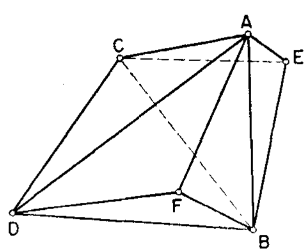

Figure 2

$$A = (1,0,1);\ B = (1,0,-1);\ C = (-1,1,0);$$

$$D = (-1,-1,0);\ E = (a,b,c);\ F = (\alpha,\beta,\gamma).$$

that E is contained in the wedge formed by the planes through ACD and BCD. Considering the sections of K by the planes $z = 0$ resp. $z = c$ it is immediate that C and E are not strictly antipodal.

Thus we may assume, for the remaining part of the proof, that K has configuration I; without loss of generality we may therefore assume that K has the form indicated in Figure 3.

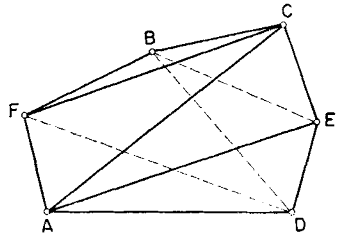

Figure 3

$$A = (-1,-1,0);\ B = (-1,1,0);\ C = (1,0,1);$$

$$D = (1,0,-1);\ E = (a,b,c);\ F = (\alpha,\beta,\gamma).$$

Since sufficiently small displacements of the points of a strictly antipodal set do not destroy strict antipodality, it follows that no generality is lost in assuming that no edge of K is parallel to a face of K, and that $|b| + |c| \neq 1$, $|\beta| + |\gamma| \neq 1$. It follows that $K^* = K + (-K)$ will have 16 triangular faces (called t-faces in the sequel), and that all other faces of K^* are parallelograms (called p-faces in the sequel). Using again a count of incidences and Euler's formula, it is easily found that K^* has 20 p-faces.

We shall end the proof of the Theorem by examining the construction of K^* and by showing that it cannot contain 20 p-faces.

Let K_1 denote the convex hull of $\{A,B,C,D,E\}$. Then K_1^* is a polyhedron with the vertices $\pm(C - A) = \pm(2,1,1)$, $\pm(D - A) = \pm(2,1,-1)$, $\pm(C - B) = \pm(2,-1,1)$, $\pm(D - B) = \pm(2,-1,-1)$, $\pm(B - A) = \pm(0,2,0)$, $\pm(C - D) = \pm(0,0,2)$, $\pm(E - A) = \pm(1 + a, 1 + b, c)$, $\pm(E - B) = \pm(1 + a, -1 + b, c)$, $\pm(E - C) = \pm(-1 + a, b, -1 + c)$, $\pm(E - D) = \pm(-1 + a, b, 1 + c)$. Since K is of type I, we have $a > 1$; this and the convexity of K_1^* imply that $|b| + |c| < 1$. (Indeed, by considering the vertices $C - A$, $D - A$, $C - B$, $D - B$, $E - A$ and $E - B$ it follows from $a > 1$ that $|c| < 1$. Then, assuming without loss of generality that $c > 0$ and $b + c > 1$, a consideration of the vertices $C - D$, $C - B$, $E - B$, $E - D$, leads to the contradiction that $C - A$ is not a vertex of K_1^*). Projecting orthogonally the part of K_1^* contained in the half-space $E^+ = \{(x,y,z)|\ x \geqq 0\}$ onto $x = 0$, we obtain a configuration of the type represented in Figure 4a. Denoting by K_2 the convex hull of $\{A,B,C,D,F\}$, the same reasoning applied to K_2^* leads to a configuration of the type given in Figure 4b.

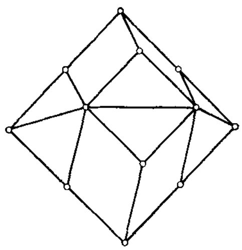

Figure 4a

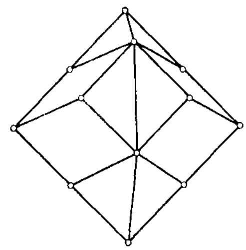

Figure 4b

Now, K^* is the convex hull of $K_1^* \cup K_2^* \cup \{E - F, F - E\}$. Obviously, $E - F$ and $F - E$ are incident only to t-faces of K^*; therefore the number of p-faces of K^* at most equals the number of p-faces in the convex hull Q of $K_1^* \cup K_2^*$. But the latter number is at most 12. Indeed, superimposing Figures 4a and 4b, we observe that every p-face of Q is a p-face of either K_1^* or K_2^*, and that only one p-face of Q contained in the half-space E^* can be incident to each of the vertices $C - A$, $D - A$, $C - B$, $D - B$.

Thus Q has at most four p-faces contained in E^+; by symmetry the same number of p-faces of Q is contained in the half-space E. Together with the four p-faces parallel to the x-axis, this yields at most 12 p-faces for Q, and thus also for K^*, in contradiction to the former assertion that K^* has 20 p-faces.

This completes the proof of the Theorem.

3. **Some related results and problems.**

i) Erdös [5] posed the problem of determining the maximal number $e(n)$ of points in E^n such that all the angles determined by triplets of the points be acute. For $n = 3$ Croft [2] recently established that $e(3) = 5$. This results also from our Theorem and from the obvious assertion $e(n) \leqq g(n)$.

ii) The inequality $e(n) \geqq 2n - 1$ was established in [3] by means of the following example (reproduced here for the sake of completeness): Let $\{e_i\}_{i=1}^n$ be mutually orthogonal unit vectors in E^n. The $(2n - 1)$-pointed set $\{A, B_2, \cdots, B_n$, $C_2, \cdots, C_n$ satisfies Erdös' condition if, e. g., $A = e_1$, $B_k = \alpha_k e_1 + e_k$, $C_k = -\alpha_k e_1 - e_k$, $k = 2, 3, \cdots, n$, where all α_k's satisfy $0 < \alpha_k < 1$ and are different from each other.

iii) As mentioned (in part) in [3], it is easily shown that $g(n)$ is also the maximal number of members in any family $\mathscr{K}$ of translates of a convex body $K \subset E^n$, provided the family satisfies any of the following conditions:

(a) The intersection of any two members of $\mathscr{K}$ is a single point;

(b) The intersection of all members of $\mathscr{K}$ is a single point, which is also the only common point of any two members of $\mathscr{K}$;

(c) The intersection of any two members of $\mathscr{K}$ is $(n - 1)$-dimensional.

The same is true if in (a) or in (c) the attention is restricted to centrally symmetric K.

iv) The restriction of $\mathscr{K}$ to families of translates of one convex body is essential in iii). This is obvious in case of conditions (a) and (b); in the case of (c) the bound is 4 for $n = 2$ (while $g(2) = 3$); already for $n = 3$ it has been proved repeatedly (e.g. by Tietze, Besicovitch, Rado; see [4] for references to these and some related results) that there exists no finite bound. In [4] it is also pointed out that arbitrarily large families $\mathscr{K}$ in E^3, any two of whose members have a 2-dimensional intersection, are obtainable as the Schlegel-diagrams of the duals of 4-dimensional "neighborly polytopes" (see [7]). This was known, however, already to Brückner [1].

Nevertheless, the following question seems to be open even for $n = 3$: How many members can a family of centrally symmetric convex bodies in E^n have, if every two have an $(n-1)$-dimensional intersection?

v) Among unsolved problems related to the Theorem of the present paper we mention:

(a) The determination of $e(n)$ and of $f(n) = g(n)$ for $n > 3$; in particular, is $e(n) = g(n)$ for all n?

(b) The determination of $h(k,n) = \max\{v(K + (-K)) \mid K \subset E^n,\ v(K) = k\}$ for $k \geqq 2n$, $n \geqq 3$.

REMARK. The example ii) above implies $h(k,n) = k(k-1)$ for $n+1 \leqq k \leqq 2n-1$; for $k > n = 2$, we have $h(k,2) = 2k$. This follows from the observation that $h(k,n) = 2s(k,n)$, where $s(k,n)$ is the maximal number of strictly antipodal pairs of vertices for a convex polyhedron $K \subset E^n$ with $v(K) = k$, and the assertion $s(k,2) = k$. To prove this latter assertion assume that $s(k,2) > k$ for some k. Let k_0 be the minimal k with this property and let K, with $v(K) = k_0$, be a k_0-gon such that more than k_0 pairs of vertices of K are strictly antipodal. Then at least one vertex V_0 is antipodal to some three consecutive vertices V_{i-1}, V_i, V_{i+1} of K; but then V_i is easily seen to be strictly antipodal only to V_0. Thus the convex hull of the $k_0 - 1$ vertices of K different from V_i yields an example showing $s(k_0 - 1, 2) > k_0 - 1$, in contradiction to the minimality of k_0.

It is worth mentioning that $s(k,3) \geqq [k/2]\cdot[k(+1)/2] + 2$ for $k \geqq 4$, the difference in behavior between $s(k, 2)$ and $s(k,3)$ being similar to the jump in the number of times the diameter of a set is assumed in 3- and 4-dimensional sets (Erdös [6]). The above inequality is easily established by placing approximately half of the points on each of two suitable circular arcs.

vi) Klee [8] defined a pair of points X_1, X_2 of a set $A \subset E^n$ to be *antipodal* provided there exist distinct parallel supporting hyperplanes H_1, H_2 of A such that $X_i \in A \cap H_i$, i $= 1,2$; he also asked about the maximal number of points in a set $A \subset E^n$ such that every two points of A are antipodal. It was established in [3] that the required number is 2^n. In analogy to the above definition of $s(k,n)$ one may ask what is the maximal number $a(k,n)$ of pairs of antipodal points in k-pointed sets in E^n. While the problem is open for $n \geqq 3$, it can be shown by arguments similar to those used above in connection with $s(k, 2)$ that $a(k, 2) = [3k/2]$.

REFERENCES

1. Brückner, M., 1909, Über die Ableitung der allgemeinen Polytope und die nach Isomorphismus verschiedenen Typen der allgemeinen Achtzelle (Oktatope), *Verh. Nederl. Akad. Wetensch.* Sect. I, **10** (1), 27pp. + 2 plates.

2. Croft, H.T., 1961, On 6-point configurations in 3-space, *J. London Math. Soc.*, **36**, 289–306.

3. Danzer, L. and Grünbaum, B., 1962, Über zwei Probleme bezüglich konvexer Körper von P. Erdös und von V.L. Klee, *Math. Z.*, **79**, 95–99.

4. Danzer, L., Grünbaum, B. and Klee, V., Helly's theorem and its relatives. *Proc. Symp. Pure Math.*, Vol. **7**, 101–180.

5. Erdös, P., 1957, Some unsolved problems, *Michigan Math. J.*, **4**, 291–300.

6. Erdös, P., 1960, On sets of distances of n points in Euclidean space, *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, **5**, 165–169.

7. Grünbaum, B. and Motzkin, T.S., On polyhedral graphs, *Proc. Symp. Pure Math.*, Vol. **7**, 285–290.

8. Klee, V., 1960, Unsolved problems in intuitive geometry, (Mimeographed notes) Seattle.

THE HEBREW UNIVERSITY OF JERUSALEM

Added in proof (June 13, 1963):

The result of Croft [2] that $e(3) = 5$ was recently established also by Schütte, K., 1963, Minimale Durchmesser endlicher Punktmengen mit vorgeschriebenem Mindestabstand, *Math. Annalen*, **150**, 91–98.